REVIEW FOR POST EXAM III MATERIAL

The final will be 10:00 AM – 11:50 AM on Monday, December 9 in CLB 202, our usual room.

You will be allowed to bring two $8\frac{1}{2} \times 11$ inch sheets of paper to the exam on which you may write anything you like; you may write on both sides. Remember that the last page of the final will be a table of Laplace transforms.

The final will cover 1.1-1.4, 2.1-2.4, 3.1-3.2, 3.4-3.5, 4.1-4.7, 4.9-4.10, 6.1-6.2, 7.1-7.6, 7.9, and 8.1-8.3. The new material is in 7.6, 7.9 and 8.1-8.3.

Note that 2.1, 3.1, 7.1, 8.1, and 8.2 did not have any homework problems assigned. The sections 2.1, 3.1, 4.1, and 7.1 did not contain any problems. 8.1 and 8.2 did contain problems, but none of them were assigned since this was background material over which you will not be explicitly tested.

DISCONTINUOUS FORCING FUNCTIONS

For an initial value problem $L[y] = g(t)$, $y(0) = y_0$, $y'(0) = y_0'$, the forcing function $g(t)$ might not be continuous or, even if it is continuous, it might be given by different formulas on different intervals. To handle this situation the *Heaviside unit step functions* $u(t-c)$ are often helpful.

$$u(t-c) = \begin{cases} 0, & t < c \\ 1, & c \le t \end{cases}$$

It is a good idea to practice graphing functions which are combinations of these step functions and other functions as well as to practice rewriting a given function as such a combination.

For example, if $a < b$, then the function which has the value k for $a \le t < b$ and is 0 otherwise can be written as $ku(t-a) - ku(t-b)$.

The Laplace transform of $u_c(t)$ is given by

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

There is also a translation property which is very useful in taking inverse Laplace transforms. Suppose $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

One now proceeds as before to transform a differential equation for $y(t)$ into an algebraic equation for $Y(s)$, find $Y(s)$, and then find $y(t)$.

THE DIRAC DELTA "FUNCTION"

This material is in section 7.9. You should refer back to the Exam III review for general information on the Laplace transform.

Sometimes the forcing function $g(t)$ in the equation $L[y] = g(t)$ represents a very large force which is applied during a very short time interval. For example, $g(t)$ might be 1000 lbs when $2.995 \leq t < 3.005$ and be 0 otherwise. Such a problem could be solved using $g(t) = 1000(u(t-2.995)-u(t-3.005))$, but it is sometimes more convenient to get an approximate solution by using the following trick.

In our example 1000 lbs is applied for 0.01 sec. The product of these two quantities is the *impulse* 10 lb-sec. Note that this is the integral of $g(t)$ over the interval $[2.995, 3.005]$, and since $g(t)$ is zero everywhere else, it is actually the integral of $g(t)$ over $-\infty < t < \infty$. Note that a lb-sec is the same as a slug-ft/sec, which means that impulse has the same units as momentum mv. In fact, the impulse causes a change in the momentum of the system. If, for example, $m = 5$ slugs, then this causes a change in velocity of 2 ft/sec.

The trick is to imagine that one could instantly change the velocity by this amount at time $t = 3$. One does this by pretending that there is a function which is zero for $t \neq 3$, is infinite for $t = 3$, and has integral over $-\infty < t < \infty$ equal to 10. This would give the same impulse to our system as our previous function $1000(u(t-2.995) - u(t-3.005))$.

To build this "function" we first imagine a "function" $\delta(t)$ which is zero for $t \neq 0$, is infinite for $t = 0$, and has integral over $-\infty < t < \infty$ equal to 1. We then translate this "function" to the right 3 units by taking $\delta(t-3)$; then this "function" is zero for $t \neq 3$, is infinite at $t = 3$, and has integral over $-\infty < t < \infty$ equal to 1. Finally we multiply this "function" by 10 to get our desired "function" $g(t) = 10\delta(t-3)$.

In order to use this we must find some way to take its Laplace transform. Suppose we have some honest function $f(t)$ and then integrate $f(t)\delta(t)$ over $-\infty < t < \infty$. Since this quantity is zero for $t \neq 0$ we ought to get the same result as integrating $f(0)\delta(t)$ over this interval, and this should give $f(0) \cdot 1 = f(0)$.

If you feel that this argument is bizzare and bogus, then you are correct. It is nonsense, but it is useful and inspired nonsense! (And it can all be made rigorous by a branch of mathematics called the theory of distributions.)

Now if we translate we get that the integral of $f(t)\delta(t-3)$ is $f(3)$. More generally, we get that the integral of $f(t)\delta(t-c)$ over $-\infty < t < \infty$ is $f(c)$. Let us specialize to $f(t) = e^{-st}$. The integral of this function over $-\infty < t < \infty$ is then e^{-sc}. If we assume that $c > 0$, then this is the same as the integral over $0 \leq t < \infty$, which is just the Laplace transform.

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}.$$

By the usual linearity property of the Laplace transform we have that $\mathcal{L}\{10\delta(t-3)\} = 10e^{-3s}$ in our example.

You can now go ahead and solve the initial value problem in the usual way by solving for $Y(s)$ and finding $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. You will usually have some terms which have to be inverted using the formula $\mathcal{L}^{-1}\{e^{-cs}G(s)\} = u(t-c)g(t-c)$, where $\mathcal{L}^{-1}\{G(s)\} = g(t)$.

SERIES SOLUTIONS

We now discuss 8.1-8.3. You should look at your class notes and the book for examples. This review just describes the general pattern of things and gives advice.

A *power series* with *center* x_0 is an expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

It has a *radius of convergence* $\rho \geq 0$ which determines an *interval of convergence* $x_0 - \rho < x < x_0 + \rho$. The series converges absolutely for x in this interval and diverges outside it. It may converge or diverge at each of the endpoints. Inside the interval of convergence you can differentiate the power series like a polynomial and multiply it by polynomials like a polynomial. So

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \text{ and } f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}.$$

Some power series have every other term equal to zero and so will have only even powers or only odd powers. In the first case you can set each even $n = 2k$ and so write $\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{k=0}^{\infty} a_{2k} (x - x_0)^{2k}$. In the second case you can set each odd $n = 2k+1$ and so write $\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{k=0}^{\infty} a_{2k+1} (x - x_0)^{2k+1}$. An important skill required for using power series to solve differential equations is reindexing. You may have a power series in which the power of $(x - x_0)$ occurring is something other than n, for example $n - 2$ or $n - 1$ or $n + 1$. To reindex a series with, say, the power $n - 1$, you can set $m = n - 1$, and so $n = m + 1$. Replace every occurrence of n in your series by $m + 1$. If the original series starts at the value $n = n_0$, then the reindexed series will start at the value $m = n_0 - 1$. Finally you will change the name of the variable back to n to finish the reindexing.

Reindexing is done when you are just about to add power series together. If you are not yet to that point in your calculations, then DON'T reindex yet. For example if you need to multiply a series by a power of $(x - x_0)$, then wait until after you have done this to reindex. Don't just automatically reindex right after taking a derivative.

Now, to solve a differential equation using power series you look for a solution of the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$. The value of x_0 to use will be given to you. Compute y' and y'' as mentioned earlier. Your differential equation should have the form

$$P(x - x_0)y'' + Q(x - x_0)y' + R(x - x_0)y = 0,$$

where P, Q, and R are polynomials in $(x - x_0)$. If it doesn't have this form as given to you, then you will have to put it in this form. For example, if you had $x_0 = 2$ and the equation $x^2y'' + y' + (1 + x)y = 0$, you would write $x = 2 + (x - 2)$, and so $x^2 = (2 + (x - 2))^2 = 4 + 4(x - 2) + (x - 2)^2$ and $1 + x = 1 + 2 + (x - 2) = 3 + (x - 2)$, so you would get

$$[4 + 4(x - 2) + (x - 2)^2]y'' + y' + [3 + (x - 2)]y = 0.$$

Plug in your series for y, y', and y''. Next do any multiplications that are necessary. This will give you a sum of several series having various powers of $(x - x_0)$ in their general terms. Reindex those series which do not have $(x - x_0)^n$ in their general terms so that they do have this power. You will now have a sum of several series which may begin at different starting values of n. Look for the highest starting value, say it is $n = N$. Find all the terms for which $n < N$ and write them out separately; this will be a finite sum of things, just a little polynomial in $(x - x_0)$. Then collect together all the remaining terms into one series which starts with $n = N$.

Next set the coefficient of each power of $(x - x_0)$ equal to zero. You will have finitely many simple equations coming from your little polynomial and then a big equation coming from your series. Solve each equation for the a_j with the biggest j. This will express a_j in terms of those a_i with $i < j$. This is called the *recurrence relation* or the *recurrence equation*. Note that some of your coefficients may be forced to have the value zero at this stage.

In general you will have two arbitrary coefficients which are the first two coefficients which are not forced to be zero. You will then be able to use the recurrence relation to inductively find all the other coefficients in terms of these two. Often you can get linearly independent solutions of the equation by first setting one of the arbitrary coefficients equal to one and the other equal to zero to get y_1, and then switching which one is zero and which is one to get y_2. You will then want to find y_1 and y_2 as follows.

Sometimes you can recognize a pattern in the coefficients a_n and get an explicit formula for them, so you can write down a formula for the entire series at once. When you are trying to do this, DON'T do the arithmetic to get some unintelligible fraction. Leave things factored out and uncancelled so you can look for things like factorials and powers.

Other times it is difficult or impossible to see a pattern and you must content yourself with finding, for example, the first four non-zero terms of y_1 and the first four non-zero terms of y_2. (Note that sometimes the series is just a polynomial, and so it may not have four non-zero terms.)

One thing that often helps you cope with the reindexing and so forth is to write out the power series as follows:

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \cdots$$

This makes it easy to differentiate and do the algebraic manipulations since you can treat it like a long polynomial. For example, you get

$$y' = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \cdots,$$

$$y'' = 2a_2 + 3 \cdot 2a_3(x - x_0) + 4 \cdot 3a_4(x - x_0)^2 + \cdots$$

Then after doing the algebraic manipulations it may be easier to see what the recurrence relation and any general pattern may be.

On the final you will be given a two part problem.

In the first part you will be given a differential equation and have to find the recurrence relation. Then STOP; that part is done. Remember that a recurrence relation contains only coefficients and numbers. It NEVER contains an x.

In the second part you will be given a recurrence relation for a series with center $x_0 = 0$ and have to find the first four non-zero terms of the series y_1 with $y_1(0) = 1$ and $y_1'(0) = 0$ (corresponding to $a_0 = 1$ and $a_1 = 0$) and the first four non-zero terms of the series y_2 with $y_2(0) = 0$ and $y_2'(0) = 1$ (corresponding to $a_0 = 0$ and $a_1 = 1$.) Remember that a_n is NOT a TERM of the series; it is only a COEFFICIENT. The term corresponding to it has the form $a_n x^n$. Your answer for the first four non-zero terms should be in the form of a polynomial with EXACTLY FOUR NON-ZERO terms. Note that the first four non-zero terms for y_1 or for y_2 might NOT be $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ because some of these coefficients could be zero.