REVIEW FOR EXAM I

This exam covers sections 1.1-1.4, 2.1-2.4, 3.1-3.2, and 3.4-3.5 in the book. This review is basically a checklist of things with which you should be familiar. As such it does not cover the material in detail. For the details you should consult your class notes, the book, and the quiz solutions posted on the course website.

TERMINOLOGY

An *ordinary differential equation* is an equation involving an independent variable, for example x, a dependent variable, for example y, and some derivatives of y with respect to x. In what follows we may sometimes use t as the independent variable and x as the dependent variable. The mathematical ideas are the same regardless of the notation.

A *solution* of the differential equation is a function g such that the equation is true when we set $y = g(x)$. Ideally you would like to have a formula for the function $g(x)$; this is called an *explicit solution*. Sometimes we cannot find such a formula but may be able to find an equation of the form $G(x, y) = C$ which relates x and y (and does not have any derivatives in it); this is called an *implicit solution*. Sometimes it may be difficult or impossible to do this. It then may be possible to obtain qualitative information about the behavior of the solutions or to get a table giving a numerical approximation to the solution for certain values of x.

The *order* of the equation is the order of the highest derivative appearing in the equation. Thus an n^{th} order equation can be written in the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

(The notation $y^{(n)}$ refers to the n^{th} derivative $\dfrac{d^n y}{dx^n}$.)

The equation is *linear* if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x).$$

We will always assume that we can solve for $y^{(n)}$ so that our equation can be written in the form

$$y^{(n)} = f(x, y, \ldots, y^{(n-1)}).$$

On this exam we will be primarily concerned with first order equations. They have the form

$$y' = f(x, y).$$

If f does not depend on y, so $y' = f(x)$, then the solution is easy to find; just integrate: $y = \int f(x)\,dx + C$.

If f does not depend on x, so $y' = f(y)$, then it is called an *autonomous* equation. These are examples of a larger family of equations called separable equations, the solution of which we will discuss later. It is also possible to get qualitative information about the solutions, another topic to be discussed later.

If a first order equation is linear it can be written in the form

$$y' + P(x)y = Q(x)$$

and can be solved in a straightforward manner we will review shortly.

Usually a first order equation, whether linear or non-linear, will have many solutions. If there is a single formula involving an unknown constant C such that every solution can be obtained by setting C equal to a number, then this formula is called a *general solution*. Some equations do not have a general solution; there will be "extra" solutions which do not arise from a general formula; these are called *singular* solutions.

Sometimes you are interested in finding a solution which has a particular value of y at a particular value of x. This is an *initial value problem* $y(x_0) = y_0$. If you have a formula, explicit or implicit, for the solution which involves a constant C, then you set $x = x_0$ and $y = y_0$ in it and solve for the constant C. Sometimes this is not enough to determine the solution; you may have to choose the sign on a square root, or the solution of the initial value problem may be a singular solution which does not follow from a general formula.

ANALYTIC METHODS

In this section we discuss methods for attempting to find formulas for the solutions of various types of first order equations. If you are given an equation you should be able to recognize what type of equation it is. Sometimes you may need to rewrite it in order to recognize its form. You should also be aware that a given equation may belong to several different types; you should be able to list them all.

1. Linear equations

   These equations have the form $y' + P(x)y = Q(x)$. Here is the procedure for solving them.

   Compute the *integrating factor* $\mu(x) = \exp(\int P(x)\, dx)$. You do NOT need to tack a $+C$ onto the integral in this expression. You should memorize this formula. Then proceed as follows.

$$\mu(x)(y' + P(x)y) = \mu(x)Q(x)$$

$$(\mu(x)y)' = \mu(x)Q(x)$$

$$\mu(x)y = \int \mu(x)Q(x)\, dx + C$$

$$y = \frac{\int \mu(x)g(x)\, dx + C}{\mu(x)}$$

   Remember that you will be working with specific functions $\mu(x)$ and $Q(x)$, so your work will not look this abstract or complicated. You should not memorize this last formula; you should learn the procedure being used.

   Note that for any linear equation you can get a general explicit solution, so if you want to solve an initial value problem $y(x_0) = y_0$ you can just set $x = x_0$ and $y = y_0$ and solve for C.

2. Separable equations

   Recall that $y' = \frac{dy}{dx}$, so our equation has the form $\frac{dy}{dx} = f(x, y)$. The equation is *separable* if it can be rewritten in the form

$$h(y)\, dy = g(x)\, dx.$$

   In this case you integrate both sides, recognizing that the integrals may differ by a constant.

$$\int h(y)\,dy = \int g(x)\,dx + C$$

This gives you an implicit solution of the form

$$G(y) = H(x) + C.$$

If possible try to solve this for y to get an explicit solution.

If you have an initial value problem set $x = x_0$ and $y = y_0$ in this equation to find C.

Warning: Sometimes these equations have "singular" solutions. This may happen if in rewriting the equation you have the possibility of dividing by zero when you divide both sides by some function of y. Examine your divisor to see for which values of y it is 0. Then check to see if any of these constant values of y gives you a solution to the original differential equation.

3. Exact equations

Suppose you can rewrite an equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

This equation is *exact* if there is a function $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$. In this case the equation has the implicit solution $F(x, y) = C$.

If the equation is exact, then we must have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. So, if $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, then the equation is NOT exact.

It turns out that if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then the equation may or may not be exact. However, if the region in the x-y plane in which M, N, $\dfrac{\partial M}{\partial y}$, and $\dfrac{\partial N}{\partial x}$ are all continuous has no "holes" in it, then the equation is exact. (The x-y plane with the origin removed has an obvious "hole" in it. The set of points (x, y) with $y > 0$ does not have any "holes" in it.)

If the equation is exact, then here is the procedure for solving it. Remember that in any specific problem you will have specific functions, and so your work will not look as abstract or complicated as what follows.

Set $\dfrac{\partial F}{\partial x} = M$.

Integrate with respect to x to get $F(x, y) = \int M\, dx + g(y) = Q(x, y) + g(y)$.

Here $g(y)$ is an as yet unknown function of y; $Q(x, y)$ is the specific function you get by integrating $M(x, y)$ with respect to x.

Take the partial derivative of F with respect to y to get $\dfrac{\partial F}{\partial y} = \dfrac{\partial Q}{\partial y} + g'(y)$.

Set $\dfrac{\partial F}{\partial y} = N$ to get $\dfrac{\partial Q}{\partial y} + g'(y) = N$.

Solve this for $g'(y) = N(x, y) - \dfrac{\partial Q}{\partial y}$.

If the equation is exact and you have done everything correctly so far this expression, when simplified, will not have any x's in it. It will be some function of y alone, so we have $g'(y) = p(y)$. If you do have any x's here, stop and check your work. Something is wrong. Do not continue until you correct the mistake.

Find $h(y) = \int p(y)\, dy$. You will not need a "$+C$" here.

Write out $F(x, y) = Q(x, y) + g(y)$.

Get your implicit solution by setting $F(x, y) = Q(x, y) + g(y) = C$.

If possible try to find an explicit solution. If you have an initial value problem try to find the C.

NUMERICAL METHODS

Suppose we are given an initial value problem

$$y' = f(x, y), \;\; y(x_0) = y_0$$

and we want to approximate the value of $y(x)$ for certain values of $x > x_0$. We might want to do this if no other method of solution is available to us or if those methods are too cumbersome or if we really just need the values of the solution at these specific numbers. There are many methods for doing this. We will limit our attention to Euler's method. There are other methods that are faster and more accurate, but they are a good bit more complicated to understand and to implement.

Suppose we have points $x_0 < x_1 < x_2 < \cdots$. We will restrict to the case in which the points are evenly spaced, so $x_{n+1} - x_n = h$, a constant. Note that this implies that $x_{n+1} = x_n + h$. Thus $x_1 = x_0 + h$, $x_2 = x_1 + h = x_0 + 2h$, $x_3 = x_2 + h = x_0 + 3h$, etc. So in general we have $x_n = x_0 + nh$.

Euler's method uses the tangent line to approximate the graph of the function. This gives the recursive formula

$$y_{n+1} = y_n + f(x_n, y_n)h.$$

It may be convenient to use the shorthand notation y'_n for the number $f(x_n, y_n)$. In this case the recursive formula would be

$$y_{n+1} = y_n + y'_n h.$$

Thus you start with the known values of x_0 and y_0, compute $y'_0 = f(x_0, y_0)$, then compute $y'_0 h$, and then compute $y_1 = y_0 + y'_0 h$. This is the first iteration of the process. Next you repeat these steps starting with x_1 and y_1. You compute $y'_1 = f(x_1, y_1)$, then $y'_1 h$, then $y_2 = y_1 + y'_1 h$. This is the second iteration of the process. You continue in this fashion until you have y_n calculated for all the given t_n.

The best way to organize this is by means of a table in which you have columns headed by n, x_n, y_n, y'_n, $y'_n h$, and y_{n+1}.

You should know how to set the problem up. If you want $y(a) = y_0$, then this means that $x_0 = a$. If the problem asks for an approximation y_m to $y(b)$, where $b > a$, this means that $x_m = b$. If you are given the value of m this will tell you that $h = (b-a)/m$. If, instead, you are given the value of h this will tell that $m = (b-a)/h$ (assuming this number is an integer). Remember that the index n we are using to count things begins at 0, not at 1 or at a, and it ends at m, not b. You should also know when to stop. If, for example, $m = 2$ give $x_2 = b$, then you should stop with y_2; this is your approximation to $y(b)$; don't compute y_3, y_4, etc.

When using your calculator you should NOT round off between computations. This can cause serious error. You can avoid rounding off by storing each new y_n value in your calculator and then recalling it when you compute $f(x_n, y_n)$ and y_{n+1}. Then store y_{n+1} and continue to the next iteration.

QUALITATIVE METHODS

Sometimes you can get a good deal of information about the general behavior of the solutions of an equation without actually solving that equation. In fact sometimes even if you know a formula for the solution it may not be easy to deduce this information from the formula. Instead we look at the function $f(x, y)$ on the right hand side of the equation $y' = f(x, y)$.

1. Direction fields

   $f(x, y)$ tells you the slope of the tangent line to the graph of a solution which passes through the point (x, y). By taking a grid of points (x, y) in the x-y plane and drawing a short line segment with slope $f(x, y)$ at each point of the grid we get a general qualitative picture of what the solutions look like, in particular their behavior as x goes to $\pm\infty$; sometimes solutions will go to $\pm\infty$, sometimes they will approach a constant y value, sometimes they may approach a straight line or a curve.

   Here are some important rules to follow:

   Suppose that you are told what grid to use. For example, you might be told to use the points (x, y) where x and y are integers between -2 and 2. There are exactly five possible values for x and five possible values for y, so you get exactly twenty-five specific points in your grid. You should make a table of values for $f(x, y)$ at these points. Do not omit any of these points. Do not add any other points. You must evaluate $f(x, y)$ and draw the line segments for precisely the values you are given. Draw the line segments so that the center of the line segment is exactly on the point (x, y); do NOT draw it with an endpoint of the line segment on the point (x, y).

   One technique that helps you find direction fields is the method of *isoclines*. For a given constant c the isocline corresponding to c is the set of all points (x, y) at which $f(x, y) = c$. At every one of these points the tangent line segment has the slope c.

   Note also that you might be given a problem where you are given a picture of a direction field and are asked questions about the behavior of the solutions as $x \to \pm\infty$.

2. Autonomous equations

The equation $y' = f(x, y)$ is *autonomous* if $f(x, y)$ does not depend on x, that is if the equation has the form $y' = f(y)$. Autonomous equations are separable and so in principle one can obtain at least an implicit solution analytically. It turns out that without doing this one can get a good bit of qualitative information, as follows.

If α_i is a number for which $f(\alpha_i) = 0$ then α_i is called a *critical point*. For each critical point α_i the constant function $y = g(x) = \alpha_i$ is a solution of the equation since $y' = \alpha_i' = 0 = f(\alpha_i) = f(y)$. Such a solution is called an *equilibrium solution* of the equation. You can find all the critical points, and from that all the equilibrium solutions, by finding all the solutions of $f(y) = 0$.

Assuming that $f(y)$ is continuous, it can change sign only at the critical points. Thus you can determine the sign of $f(y)$ and hence the sign of the slope y' by evaluating $f(y)$ at a point in each of the intervals into which the critical points chop the y axis. This will tell you whether the solutions between the equilibrium solutions are increasing or decreasing.

An equilibrium solution is *stable* if the solutions immediately above and below it approach the equilibrium solution as $x \to \infty$. This will happen if solutions immediately above are decreasing and those immediately below are increasing.

An equilibrium solution is *unstable* if the solutions immediately above and below it move away from the equilibrium solution as $x \to \infty$. This will happen if solutions immediately above are increasing and those immediately below are decreasing.

An equilibrium solution is *semistable* if the solutions immediately on one side approach the equilibrium solution and those immediately on the other side move away from the equilibrium solution as $x \to \infty$. This will happen if solutions on both sides are increasing or solutions on both sides are decreasing.

3. Existence and uniqueness

Given an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ one can ask three questions:

(1) Does a solution $y = g(x)$ exist?

(2) Is the solution unique, i.e. is there only one solution?

(3) For what values of x does the solution exist?

The answers to these questions may take different forms depending on whether the equation is linear or non-linear.

For a linear equation $y' + P(x)y = Q(x)$, $y(x_0) = y_0$

(1) A solution will exist for those x_0 such that both P and Q are continuous at x_0. (It is customary to express the set of such x_0 as unions of open intervals, so we might have an answer like $(0, 1) \cup (2, \infty)$, which could also be written as $0 < x < 1$ or $x > 2$. Note that even if P and Q were continuous from the right at 2 we would still exclude 2 from our interval.)

(2) The solution in (1) is unique.

(3) If both P and Q are continuous in the interval $a < x < b$, where $a < x_0 < b$, then $g(x)$ exists for all x in the interval.

Note that if you are given a *specific* IVP, say $y(9) = 8$, and are asked to find the largest open interval on which the solution to this IVP exists, then you will *choose the largest interval* $\alpha < x < \beta$ such that (1) $P(x)$ *and* $Q(x)$ *are continuous on this interval* and (2) $x_0 = 9$ *is in this interval.* Ignore all the other intervals which do NOT contain $x_0 = 9$.

If we have an equation $y' = f(x, y)$, $y(x_0) = y_0$ which is not necessarily linear then

(1) A solution will exist for those (x_0, y_0) at which f is continuous.

(2) The solution in (1) is unique if $\dfrac{\partial f}{\partial y}$ is continuous at (x_0, y_0).

(3) There will be some number $h > 0$ such that a solution exists for $x_0 - h < x < x_0 + h$.

(Note that when we talk about the region where f or $\frac{\partial f}{\partial y}$ is continuous we are talking about an "open set" where these functions are continuous; this means that there is a little open rectangle $a < x < b$, $c < y < d$ such that these functions are continuous in the rectangle and $a < x_0 < b$, $c < y_0 < d$.)

APPLICATIONS

There are many different applications of differential equations to physics, chemistry, engineering, finance, population dynamics, and other areas, so it is possible only to give some general advice here on how to proceed.

First, read the problem very carefully. Then read it again. And again. Be sure to pay close attention to what the words mean and how they are used. Do not ignore things the problem says. Do not read into the problem things that the problem does not say.

Identify the independent variable. Is it time? position? velocity? something else?

Identify the dependent variable. Is it position? velocity? temperature? mass? money? population? something else?

In what units are each of these variables being measured? seconds? feet? kilometers? degrees Celsius? grams? pounds? volts? people? something else?

What basic law or principle governs the problem? Newton's second law of motion? Newton's law of cooling? Malthus' law of population growth? the logistic law of population growth? Ohm's law of resistance? common sense? something else? Often a careful reading of the problem will reveal what the law or principle is.

Express the law or principle as a differential equation involving the two variables. Somewhere you should have the derivative of the dependent variable with respect to the independent variable. This should be expressed in terms of the two variables and perhaps certain constants that you deduce from the statement of the problem.

Make sure that your units match up. You should not have grams/liter equaling liters/minute, for example. Thinking through how the units cancel and match up is often a good guide to what the equation ought to be.

Once you get the differential equation read the problem again to see what it wants you to do with it. Should you just write it down and stop? Should you solve it? Should you find values for the variables or constants such that certain conditions are satisfied?