Calculus II (CRN 64589) – Final Exam Sample Problems

The final exam will have ten questions.

I. Integration by Substitution

I-1. Evaluate $\int \frac{\cos(\theta)}{1+\sin^2(\theta)}\,d\theta$.

I-2. Evaluate $\int \frac{x^2}{\sqrt{1-x}}\,dx$.

I-3. Evaluate $\int \frac{e^{-x}}{1+e^{-x}}\,dx$.

I-4. Evaluate $\int \frac{\left(\ln(x)\right)^2}{x}\,dx$.

I-5. Use the substitution $x = \sin^2(\theta)$ with $0 \leq \theta \leq \frac{\pi}{2}$ to express $\int_0^{1/2} \sqrt{\frac{x}{1-x}}\,dx$ as a trigonometric integral.

I-6. Use the substitution $u = 1+\sqrt{x}$ to evaluate $\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$.

II. Integration by Parts

II-1. Evaluate $\int_0^{\pi/2} \theta\cos(\theta)\,d\theta$.

II-2. Evaluate $\int xe^{-2x}\,dx$.

II-3. Evaluate $\int x^3\ln(x)\,dx$.

II-4. Evaluate $\int_0^1 \arctan(x)\,dx$.

II-5. Evaluate $\int \frac{\ln(1+x^2)}{x^2}\,dx$.

III. Trigonometric Integrals

III-1. Evaluate $\int \cos^3(\psi)\,d\psi$.

III-2. Evaluate $\int_{\pi/8}^{\pi/4} \cos^2(2\theta)\, d\theta$.

III-3. Evaluate $\int \frac{1}{1+\sin(\theta)}\, d\theta$. (Hint: Simplify the integrand by multiplying by $1-\sin(\theta)$ in the numerator and denominator.)

III-4. Evaluate $\int \sec^4(x)\, dx$.

III-5. Evaluate $\int \tan^3(x)\, dx$.

IV. Partial Fractions

IV-1. Find the partial fraction expansion of $\frac{2x}{(x+1)(2x+1)(3x+1)}$.

IV-2. Find the partial fraction expansion of $\frac{3x+1}{x^2(x-3)}$.

IV-3. Find the partial fraction expansion of $\frac{x^2}{(x+1)(x^2+4)}$.

IV-4. Find the partial fraction expansion of $\frac{1}{(x^2-1)^2}$.

IV-5. Find the partial fraction expansion of $\frac{1}{(x^2+1)(x^2+2)}$.

V. Improper Integrals

V-1. Evaluate $\int_0^{\infty} xe^{-x^2}\, dx$. (Hint: Use integration by substitution to evaluate the antiderivative.)

V-2. Evaluate $\int_0^{\infty} \frac{2x}{(x+1)(2x+1)(3x+1)}\, dx$. (Hint: See Problem IV-1.)

V-3. Evaluate $\int_1^{\infty} \frac{\ln(1+x^2)}{x^2}\, dx$. (Hint: See Problem II-5.)

V-4. Evaluate $\int_0^1 \sqrt{\frac{x}{1-x}}\, dx$. (Hint: See Problem I-6.)

V-5. Use a comparison to decide whether the integral $\int_0^{\infty} \frac{x^2}{x^4+1}\, dx$ converges or diverges.

V-6. Use a comparison to decide whether the integral $\int_1^{\infty} \frac{2+\cos(3x)}{x}\, dx$ converges or diverges.

VI. Geometric Series

VI-1. Does the series $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{2n}}$ converge or diverge? If it converges, what is its value?

VI-2. What condition on x is required for the series $\sum_{n=0}^{\infty} e^{-nx}$ to converge? When this condition is met, what is the value of the series?

VI-3. Use a geometric series to evaluate the repeating decimal $0.2\overline{31} = 0.2313131\ldots$ as a rational number.

VI-4. Write $\frac{x}{3x-2}$ as a power series with center -1 and state the Interval of Convergence of the series.

VI-5. Write the series

$$\frac{1}{t} - \frac{1}{t^3} + \frac{1}{t^5} - \frac{1}{t^7} + \ldots$$

using sigma notation with the sum starting at $n = 1$. State the condition required for the series to converge and the value of the series when it does converge.

VI-6. Write the series

$$\left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{8} + \frac{1}{27}\right) + \left(\frac{1}{16} - \frac{1}{81}\right) + \left(\frac{1}{32} + \frac{1}{243}\right) + \ldots$$

using sigma notation with the sum starting at $n = 1$. What is the value of the series?

VII. Taylor Polynomials/Series

VII-1. Let f be a function whose n^{th} derivative at 1 is $f^{(n)}(1) = (-1)^n(n+2)!$ for $n \geq 0$. What is the Taylor series of f with center 1?

VII-2. Find the third Taylor polynomial $T_3(x)$ for $f(x) = \sin(x)$ with center $\frac{\pi}{4}$.

VII-3. Continuing with Problem VII-2, estimate the error in using $T_3(x)$ to approximate $\sin(x)$ on the interval $[\pi/4, \pi/2]$.

VII-4. Find the second Taylor polynomial $T_2(x)$ for $f(x) = \frac{1}{\sqrt{x}}$ with center 9.

VII-5. Continuing with Problem VII-4, estimate the error in using $T_2(x)$ to approximate $\frac{1}{\sqrt{x}}$ on the interval $[8, 9]$.

VII-6. Find a value $c > 1$ such that we may approximate $f(x) = \ln(x)$ by its second Taylor polynomial $T_2(x)$ with center 1 on the interval $[1, c]$ with error no more than 9×10^{-3}.

VIII. Interval of Convergence

VIII-1. Find the Interval of Convergence of $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!(n+1)!}(x-1)^n$.

VIII-2. Find the Interval of Convergence of $\sum_{n=0}^{\infty} (n!)^2 x^{2n}$.

VIII-3. Find the Interval of Convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n + 2^n}(x+2)^{3n}$.

VIII-4. Find the Interval of Convergence of $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} x^n$.

VIII-5. Find the Interval of Convergence of $\sum_{n=0}^{\infty} \frac{1}{n+3}(x-3)^n$.

VIII-6. Find the Interval of Convergence of $\sum_{n=0}^{\infty} \sqrt{2n+3}\,(2x+3)^{2n}$.

IX. Differentiate/Integrate/Manipulate Power Series

IX-1. Given that $(1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$ for $-1 < x < 1$, write the Maclaurin series for $\frac{1}{\sqrt{1-x^2}}$.

IX-2. Use your answer to Problem IX-1 to find the Maclaurin series for $\arcsin(x)$. Write the coefficient of x^9 in this series as an explicit rational number.

IX-3. Express the integral $\int_0^1 \frac{\sin(x)}{x}\,dx$ as a series. Use your answer to find the value of the integral correct to three decimal places.

IX-4. Rewrite the series $\sum_{n=0}^{\infty} n(n-1)2^n(x-1)^{n-2}$ as power series whose general term involves $(x-1)^n$.

IX-5. Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} x^n$. Express $xf'(x)$ as a power series.

IX-6. Express $\int_0^{1/2} \frac{e^x - 1}{x}\,dx$ as a series.

X. Parameterizations/Polar Coordinates

X-1. Write a parameterization of the ellipse $4(x+1)^2 + y^2 = 16$. Sketch the ellipse.

X-2. Write a parameterization of the curve $x = \cos(y)$.

X-3. Eliminate the parameter in the parameterization

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}$$

and use this to sketch the trajectory of this parameterization.

X-4. Find the arc length of the segment of the curve given in Problem X-3 from the point where $t = 0$ to the point where $t = 1$.

X-5. Find the equation of the tangent line to the curve parameterized by

$$\begin{cases} x = 2 - te^{-t} \\ y = t + e^t \end{cases}$$

at the point where $t = 0$.

X-6. Find two sets of polar coordinates for the point $(x, y) = (-1, 1)$, one with $r > 0$ and the other with $r < 0$.

X-7. What is the equation of the curve $r = \sin(\theta) + \cos(\theta)$ in rectangular coordinates? Use this equation to sketch the curve.

X-8. Write an equation for the line $x + y = 1$ in polar coordinates.

X-9. What is the equation of the curve $r = \sin(2\theta)$ in rectangular coordinates?

X-10. Write an equation for the hyperbola $xy = 1$ in polar coordinates.