PROBLEM LIST
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Contents

1. Sums of dynamical heights on the projective line (Fili)

Posed by: Paul Fili, Oklahoma State University

Suppose $f(z), g(z) \in \overline{\mathbb{Q}}(z)$ are rational maps with algebraic coefficients. The *Arakelov-Zhang pairing*, introduced by Petsche, Szpiro, and Tucker [PST12] and denoted $\langle f, g\rangle_{\mathrm{AZ}}$ measures a sort of dynamical distance between the two maps. It has the property that $\langle f, g\rangle_{\mathrm{AZ}} \geq 0$ for all rational maps $f, g \in \overline{\mathbb{Q}}(z)$ and $\langle f, g\rangle_{\mathrm{AZ}} = 0$ if and only if $\mathrm{PrePer}(f) = \mathrm{PrePer}(g)$ (equivalently, if $\mathrm{PrePer}(f) \cap \mathrm{PrePer}(g)$ is infinite; see [PST12, Theorem 3]). The Arakelov-Zhang pairing is defined analytically in terms of the canonical adelic measures associated to each map, and has the property that if α_n is any sequence of small points for f, that is, $h_f(\alpha_n) \to 0$ and the α_n are distinct, then

$$h_g(\alpha_n) \to \langle f, g\rangle_{\mathrm{AZ}}.$$

Remarkably, this property is symmetric, and if β_n is a sequence of small points for g, then

$$h_f(\beta_n) \to \langle f, g\rangle_{\mathrm{AZ}}$$

as well.

It follows that, if $\langle f, g\rangle_{\mathrm{AZ}} > 0$, then

$$\liminf_{\alpha\in\overline{\mathbb{Q}}} h_f(\alpha) + h_g(\alpha) > 0,$$

and further, by taking a sequence of small points for either f or g, one can see that in fact,

$$\liminf_{\alpha\in\overline{\mathbb{Q}}} h_f(\alpha) + h_g(\alpha) \leq \langle f, g\rangle_{\mathrm{AZ}}.$$

We can think of this sum of heights as essentially the natural height associated to the split map $f \times g : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. As N.M. Mavraki pointed out to the author, it follows from a theorem of Zhang [Zha95, Theorem 1.10] that

$$\liminf_{\alpha\in\overline{\mathbb{Q}}} h_f(\alpha) + h_g(\alpha) \geq \frac{1}{2}\langle f, g\rangle_{\mathrm{AZ}}.$$

This result inspires our question:

Question 1.1. *For two rational maps $f(z), g(z) \in \overline{\mathbb{Q}}(z)$ with $\langle f, g\rangle_{\mathrm{AZ}} > 0$, what is the value of*

$$\liminf_{\alpha\in\overline{\mathbb{Q}}} h_f(\alpha) + h_g(\alpha)?$$

By the above, the value must lie between $\frac{1}{2}\langle f, g\rangle_{\mathrm{AZ}}$ and $\langle f, g\rangle_{\mathrm{AZ}}$. However, the author is unaware of any examples where the lower bound of Zhang seems to be achieved, and in some cases, it is impossible to achieve. For example, consider the case of $f(z) = z^2$ and $g(z) = 1 - (1 - z)^2$, which is f conjugated by the map $1 - z$. It is easy to see that $h_f(\alpha) = h(\alpha)$, $h_g(\alpha) = h(1 - \alpha)$, and so

$$h_f(\alpha) + h_g(\alpha) = h(\alpha) + h(1 - \alpha)$$

a quantity famously studied by Zhang [Zha92] and Zagier [Zag93]. In fact, Zagier was able to give an explicit result, namely, that

$$h(\alpha) + h(1-\alpha) \geq \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406059\ldots$$

for all $\alpha \neq 0, 1, (1 \pm \sqrt{-3})/2$, with equality if and only if α or $1-\alpha$ is a primitive 10th root of unity. However, by the work of Petsche, Szpiro, and Tucker [PST12, Prop. 18], we know that in the case of $f(z) = z^2$ and $g(z) = 1-(1-z)^2$,

$$\langle f, g\rangle_{\mathrm{AZ}} = \int_0^1 \log|1+e^{2\pi i t}|\,dt = \frac{3\sqrt{3}}{4\pi}L(2,\chi) = 0.323067\ldots$$

where χ is the nontrivial quadratic character modulo 3. It follows that

$$h(\alpha) + h(1-\alpha) \geq \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406059\ldots > \frac{1}{2}\langle f, g\rangle_{\mathrm{AZ}} = 0.161533\ldots$$

for all but finitely many $\alpha \in \overline{\mathbb{Q}}$. In particular, in this example,

$$\liminf_{\alpha\in\overline{\mathbb{Q}}} h_f(\alpha) + h_g(\alpha) > \frac{1}{2}\langle f, g\rangle_{\mathrm{AZ}}.$$

Zagier speculated that a spectrum of such height values might be found, leading to a smallest limit point for the height in this example. It seems quite plausible that as the spectrum is determined, that the limit infimum may in fact be the value of the Arakelov-Zhang pairing itself. I would make the following conjecture:

Conjecture 1.2. *For two rational maps $f(z), g(z) \in \overline{\mathbb{Q}}(z)$, we have*

$$\liminf_{\alpha\in\overline{\mathbb{Q}}} h_f(\alpha) + h_g(\alpha) = \langle f, g\rangle_{\mathrm{AZ}}.$$

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2. Mahler measure and composition of polynomials (Granville)

Posed by: Andrew Granville, University of Montreal

The Mahler measure of a polynomial $f(z) = c_d \prod_{k=1}^{d}(z - z_k) \in \mathbb{C}[z]$, with $c_d \neq 0$, is given by

$$M(f) := \exp\left(\frac{1}{2\pi}\int \log|f(e^{i\theta})|\,d\theta\right) = |c_d|\prod_{k=1}^{d}\max(1, |z_k|). \tag{2.1}$$

Question 2.1. *Let $f, g \in \mathbb{C}[z]$ be two polynomials, and let $f \circ g$ be their composition. What is the relation between $M(f \circ g)$ and $M(f), M(g)$? Can one prove effective (and sharp) estimates for $M(f \circ g)$ in terms of $M(f), M(g)$?*

Question 2.2. *How does the Mahler measure behave in the context of polynomial dynamics? Let $f \in \mathbb{C}[z]$, $\deg(f) \geq 2$, and consider the n-fold iterated composition of f denoted by f^n. What is the asymptotic behavior of $M(f^n)$?*

3. Variation of heights for families of curves (Mavraki)

Posed by: Niki Myrto Mavraki, Harvard University

3.1. **Setting and background.** Let B be an irreducible projective curve and let

$$f, g : B \times \mathbb{P}^1 \to \mathbb{P}^1$$

be meromorphic maps. We write $f_\lambda = f(\lambda, \cdot)$ (respectively g_λ) for each $\lambda \in B$. Note that each f_λ is a rational map and that for all but finitely many $\lambda \in B$ the maps have equal degree. For example we could have $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be given by the degree 2 map $p(\lambda, z) = \lambda z^2 + 1$ for each $\lambda \in \mathbb{C} \setminus \{0\}$, whereas $p(0, z) = 0$ and $p(\infty, z) = \infty$ are constant maps. The map f can also be seen as a single rational function $\mathbf{f} \in \mathbb{C}(B)(z)$ defined over the function field $k = \mathbb{C}(B)$. In our example $\mathbf{p}(z) = tz^2 + 1$ defined over $\mathbb{C}(t) = \mathbb{C}(\mathbb{P}^1)$. Throughout this note we will assume that

$$\deg \mathbf{f} \geq 2,$$

so that also

$$\deg f_\lambda = \deg \mathbf{f} \geq 2$$

for all but finitely many $\lambda \in B$. We will restrict our attention to maps f and curves B that are defined over $\overline{\mathbb{Q}}$, so that for each $\lambda \in B(\overline{\mathbb{Q}})$ except finitely many we have a canonical height $\hat{h}_{f_\lambda} : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ associated to $f_\lambda \in \overline{\mathbb{Q}}(z)$ as in [CS]. This canonical height appears also in Zhang's work [Zh1]. It is the height associated with the f_λ-invariant metrized line bundle $\overline{L}_{f_\lambda}$, with underlying line bundle $L = (\infty)$. We refer the reader to Zhang's article for the construction of this line bundle for an arbitrary polarized dynamical system. Similarly, for the 'geometric' map $\mathbf{f}$ we have a 'functional' canonical height $\hat{h}_{\mathbf{f}} : \mathbb{P}^1(\overline{k}) \to \mathbb{R}$ (using the places of the function field k), which is also induced by an $\mathbf{f}$-invariant metric $\overline{L}_{\mathbf{f}}$.

Zhang [Zh1, Theorem 1.4], Gubler [Gu1, Gu2, Gu3] and Chambert-Loir–Thuillier [CLT] developed an intersection theory for certain adelic metrized line bundles called integrable. In particular, for all but finitely many $\lambda \in B(\overline{\mathbb{Q}})$ we have a non-negative intersection number

$$\overline{L}_{f_\lambda} \cdot \overline{L}_{g_\lambda} \geq 0$$

and equality holds if and only if the associated canonical height functions agree

$$\hat{h}_{f_\lambda} \equiv \hat{h}_{g_\lambda}.$$

Similarly, $\overline{L}_{\mathbf{f}} \cdot \overline{L}_{\mathbf{g}} \geq 0$ with equality if and only if the functional heights associated with $\mathbf{f}$ and $\mathbf{g}$ agree. It is perhaps helpful to note that by the works [BR, CLT, FRL, PST] we know that for any infinite sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ with $\hat{h}_{f_\lambda}(x_n) \to 0$, we have $\hat{h}_{g_\lambda}(x_n) \to \overline{L}_{f_\lambda} \cdot \overline{L}_{g_\lambda}$, a result tightly connected with arithmetic equidistribution. We point out that the Arakelov–Zhang pairing $\overline{L}_{f_\lambda} \cdot \overline{L}_{g_\lambda}$ can also be though of as a global mutual energy pairing in the potential theoretic viewpoint of [BR, FRL]. We have

$$\overline{L}_{f_\lambda} \cdot \overline{L}_{g_\lambda} = \frac{1}{2}((\mu_f - \mu_g, \mu_f - \mu_g)),$$

where μ_f and μ_g are the adelic canonical measures associated to f and g and $((\cdot,\cdot))$ is a sum of local mutual energies; see [FRL, Fi].

3.2. **Questions and state of affairs.** One can ask how the energy pairings $\overline{L}_{f_\lambda}\cdot\overline{L}_{g_\lambda}$ behave as a function of λ.

Question 3.1. *Let B be an irreducible projective curve defined over $\overline{\mathbb{Q}}$ and let $f,g : B\times\mathbb{P}^1 \to \mathbb{P}^1$ be holomorphic maps also defined over $\overline{\mathbb{Q}}$. Let $h : B(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a Weil height on B corresponding to a divisor with degree 1. Is it true that*

$$\lim_{h(\lambda)\to\infty}\frac{\overline{L}_{f_\lambda}\cdot\overline{L}_{g_\lambda}}{h(\lambda)} = \overline{L}_{\mathbf{f}}\cdot\overline{L}_{\mathbf{g}}? \tag{3.1}$$

To my knowledge, there is no known answer to Question 3.1 in any example; it is unclear to me what the answer should be. That said, in recent work with Schmidt [MS], we get in particular that the following holds.

Theorem 3.2 ([MS])**.** *In the setting of Question 3.1, there is a computable constant $m := m_{\mathbf{f},\mathbf{g}} > 0$ such that*

$$\liminf_{h(\lambda)\to\infty}\frac{\overline{L}_{f_\lambda}\cdot\overline{L}_{g_\lambda}}{h(\lambda)} \geq \frac{\overline{L}_{\mathbf{f}}\cdot\overline{L}_{\mathbf{g}}}{m}. \tag{3.2}$$

The reader should compare this result with Yuan–Zhang's work [YZ], where qualitative analogs of Theorem 3.2 are obtained in much broader generality.

In absence of an answer to Question 3.1 it may be premature to ask if one can hope for a much better behavior of the energy pairing function. The following question is inspired by an analogous one of Call–Silverman in some sense; see §3.3.

Question 3.3. *In the setting of Question 3.1, is there an $\mathbb{R}$-divisor $D = D_{\mathbf{f},\mathbf{g}}$ with degree $\overline{L}_{\mathbf{f}}\cdot\overline{L}_{\mathbf{g}}$ so that the function $\lambda\mapsto\overline{L}_{f_\lambda}\cdot\overline{L}_{g_\lambda}$ for $\lambda\in B(\overline{\mathbb{Q}})$ with finitely many exceptions, is a Weil height on B corresponding to D up to a bounded error term? In other words, do we have*

$$\overline{L}_{f_\lambda}\cdot\overline{L}_{g_\lambda} = h_{D_{\mathbf{f},\mathbf{g}}}(\lambda) + O(1)? \tag{3.3}$$

It is not a priori clear, whether (3.1) or (3.3) can hold even in the case that $\overline{L}_{\mathbf{f}}\cdot\overline{L}_{\mathbf{g}} = 0$. However, the geometric Bogomolov-type result [MS, Theorem 4.1] I recently established with Schmidt implies that this is indeed the case. Since this is a non-trivial statement (at least with the proof we currently have) we record it here as a theorem.

Theorem 3.4 ([MS])**.** *If $\overline{L}_{\mathbf{f}}\cdot\overline{L}_{\mathbf{g}} = 0$, then Questions 3.3 and 3.1 have an affirmative answer.*

3.3. **Motivation and more questions.** The adelic intersection pairings can be used to extend the definition of canonical heights from points to subvarieties. For example, looking at the split geometric (defined over k) map $\mathbf{\Phi} = (\mathbf{f},\mathbf{g}) : \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1\times\mathbb{P}^1$ acting as $(\mathbf{f},\mathbf{g})(x,y) = (\mathbf{f}(x),\mathbf{g}(y))$, we may think of $\overline{L}_{\mathbf{f}}\cdot\overline{L}_{\mathbf{g}}$ as the canonical height of the diagonal

$\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ under the action of $(\mathbf{f}, \mathbf{g})$. Concretely, $\hat{h}_{\mathbf{\Phi}}(\Delta) = \overline{L}_{\mathbf{f}} \cdot \overline{L}_{\mathbf{g}}$. Similarly, if $\Phi_\lambda = (f_\lambda, g_\lambda)$, for $\lambda \in B(\overline{\mathbb{Q}})$, then we have $\hat{h}_{\Phi_\lambda}(\Delta) = \overline{L}_{f_\lambda} \cdot \overline{L}_{g_\lambda}$.

With the notion of such heights at hand, one can generalize Questions 3.1 and 3.5. We adopt this perspective in order to relate our questions with previous results and questions by Call, Silverman and others which highly guided our speculations.

For instance one can replace the diagonal curve by any curve $\mathbf{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ defined over k. Note that $\mathbf{C}$ then induces a family of curves $\{C_t\}_{t \in B'}$ where B' is a Zariski open subset of B. For instance the curve $x = ty$ defined over $\mathbb{C}(t) = \mathbb{C}(\mathbb{P}^1)$ yields a family of lines $\{x = \lambda y\}_{\lambda \in \mathbb{P}^1(\mathbb{C})} \subset \mathbb{P}^1 \times \mathbb{P}^1$. Recall our assumption that B is defined over $\overline{\mathbb{Q}}$.

Question 3.5. *Let $\mathbf{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a strict subvariety and let $\mathbf{\Phi} = (\mathbf{f}, \mathbf{g})$ be a split map such that $\deg \mathbf{f} = \deg \mathbf{g} \geq 2$, all defined over $\overline{\mathbb{Q}}(B)$. Let h be a Weil height on B corresponding to a divisor with degree 1. Do we have*

$$\lim_{h(\lambda) \to \infty} \frac{\hat{h}_{\Phi_\lambda}(C_\lambda)}{h(\lambda)} = \hat{h}_{\mathbf{\Phi}}(\mathbf{C})? \tag{3.4}$$

If $\mathbf{C}$ is a point (in $\mathbb{P}^1 \times \mathbb{P}^1$), then the question has an affirmative answer as is established by Call–Silverman [CS], who generalized analogous results in the setting of abelian varieties. They further asked whether, in the case of points, the following stronger result is true.

Question 3.6. *In the setting of Question 3.4 is there an $\mathbb{R}$-divisor $D = D_{\mathbf{\Phi}, \mathbf{C}}$ with degree $\hat{h}_{\mathbf{\Phi}}(\mathbf{C}) \in \mathbb{R}$ so that*

$$\hat{h}_{\Phi_\lambda}(C_\lambda) = h_{D_{\mathbf{\Phi}, \mathbf{C}}}(\lambda) + O(1)? \tag{3.5}$$

Even in the case of points, this question remains open in general. The two more general results in this direction are obtained by Ingram [In] and later also by Favre–Gauthier [FG], who established its validity for polynomials $\mathbf{f}, \mathbf{g}$ and points $\mathbf{C}$ and by DeMarco–Mavraki [DM] who allowed for arbitrary rational maps $\mathbf{f}, \mathbf{g}$ but imposed a dynamical condition on the point $\mathbf{C}$. We refer the reader to [DM] and the references therein for a current overview of the status of this problem and the difficulties that arise. If $\mathbf{C}$ is a curve on the other hand, very little is known such as the analogs of Theorems 3.2 and 3.4 shown in [MS].

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4. Dynamical Mordell–Lang for semiconjugacies (O'Desky)

Posed by: Andrew O'Desky, Princeton University

4.1. **Introduction.** Let X be a smooth projective complex variety. Let Φ be a self-map of X. Fix a point $x \in X$. For a subvariety $Y \subset X$ let $\mathcal{I}_\Phi(x, Y) \coloneqq \{n \in \mathbb{N} : \Phi^{\circ n}(x) \subset Y\}$. Let X' be another smooth projective complex variety with self-map Φ'. Let $\alpha\colon X \to X'$ be a morphism satisfying $\alpha\Phi = \Phi'\alpha$ (α is a semiconjugacy). Then one has the containment $\mathcal{I}_\Phi(x, Y) \subset \mathcal{I}_{\Phi'}(\alpha x, \alpha Y)$, and we let

$$\mathcal{E}_\alpha(x, Y) \coloneqq \mathcal{I}_{\Phi'}(\alpha x, \alpha Y) - \mathcal{I}_\Phi(x, Y).$$

This set is large when there are many indices n such that $\Phi'^{\circ n}(\alpha x) \in \alpha Y$ but $\Phi^{\circ n}(x) \notin Y$. For instance, this does not occur if α is injective, in which case $\mathcal{E}_\alpha(x, Y)$ is empty.

Conjecture (relative Dynamical Mordell–Lang). *$\mathcal{E}_\alpha(x, Y)$ is a finite union of arithmetic progressions.*

When α is constant this reduces to Dynamical Mordell–Lang (DML) for (X, Φ, x, Y), so this may be seen as a relative reformulation of the (cyclic) Dynamical Mordell–Lang Conjecture.

For short, call α "dynamical" if $\mathcal{E}_\alpha(x, Y)$ is a finite union of arithmetic progressions for all x and Y. Let

$$\begin{tikzcd} (X, \Phi) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & (X', \Phi') \arrow[dl, "\gamma"] \\ & (X'', \Phi'') & \end{tikzcd}$$

be a commutative diagram of semiconjugacies. The following facts are easy to show:

1. $\mathcal{E}_\beta(x, Y)$ is the disjoint union of $\mathcal{E}_\alpha(x, Y)$ and $\mathcal{E}_\gamma(\alpha x, \alpha Y)$.
2. If each of α, γ (resp. β, γ) is dynamical, then β (resp. α) is dynamical.
3. If each of α, β is dynamical and α is surjective, then γ is dynamical.

In particular, if β and γ are constant and α is surjective and dynamical, then (X, Φ) satisfies DML if and only if (X', Φ') satisfies DML. For instance, if X has Kodaira dimension zero then the Albanese morphism α is surjective with connected fibers, and then DML for (X, Φ) holds if and only if the Albanese morphism α of X is dynamical.

The following observation was worked out in conversations with Thomas Tucker.

Proposition 4.1. *Finite semiconjugacies satisfy relative DML.*

Proof. Let $\alpha\colon (X, \Phi) \to (X', \Phi')$ be a semiconjugacy which is a finite morphism. To show α is dynamical it suffices to show that (X, Φ) satisfies DML if and only if (X', Φ') satisfies DML. For the 'only if' direction, observe that $\mathcal{I}_\Phi(x, \alpha^{-1}Z) = \mathcal{I}_{\Phi'}(y, Z)$ where $x \in \alpha^{-1}y$ is any preimage, since α is surjective. For the 'if' direction, let $Y \subset X$ be a closed subvariety and let $x \in X$. Write $Z = \alpha Y$ and decompose $\alpha^{-1}Z = W_1 \cup \cdots \cup W_r$ into irreducible closed subvarieties W_i, none containing another. If $i \neq j$ then $\dim W_i \cap W_j < \min\{\dim W_i, \dim W_j\}$, so by induction on dimension we may suppose that $I \coloneqq \cup_{i\neq j}\mathcal{I}_\Phi(x, W_i \cap W_j)$ is a finite union of arithmetic progressions. By assumption, the set $\mathcal{I}_{\Phi'}(\alpha x, Z) = \mathcal{I}_\Phi(x, \alpha^{-1}Z) = \cup_k \mathcal{I}_\Phi(x, W_k)$ is a finite union of arithmetic progressions. It is known that $\mathcal{I}_\Phi(x, W_k) = A_k \sqcup B_k$ where A_k is a finite union of arithmetic progressions and B_k is a set of Banach density zero [BGT16, Thm. 11.1.0.7]. It follows that $\mathcal{I}_\Phi(x, W_k)\backslash(I \cap \mathcal{I}_\Phi(x, W_k))$ can also be expressed as a disjoint union of A'_k, a finite union of arithmetic progressions, with B'_k, a set of Banach density zero. The equality

$$\mathcal{I}_\Phi(x, \alpha^{-1}Z)\backslash I = \sqcup_k \big(\mathcal{I}_\Phi(x, W_k)\backslash(I \cap \mathcal{I}_\Phi(x, W_k))\big) = \sqcup_k (A'_k \sqcup B'_k)$$

shows that $\sqcup_k B'_k$ is a finite union of arithmetic progressions, and therefore must be a finite set. It follows that $\mathcal{I}_\Phi(x, W_k)$ is a finite union of arithmetic progressions for all k. Since α is finite, the closed subvariety Y is a union of some subset of the irreducible components $\{W_1, \ldots, W_k\}$ and therefore $\mathcal{I}_\Phi(x, Y)$ is a finite union of arithmetic progressions as well. $\square$

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5. Equidistribution of integer sequences in $\mathbb{Z}_p$ (Somesunderam, O'Dorney)

Posed by: Naveen Somasunderam, SUNY Plattsburgh, with an additional question by Evan O'Dorney, Notre Dame

5.1. Introduction.

We look at the equidistribution properties of integer sequences over the p-adic ball $\mathbb{Z}_p$. Some interesting sequences to study are linear recurrence sequences and p-adic log sequences $c_1 \log_p(r_1) + c_2 \log_p(r_2)$ where $\log_p$ is the p-adic logarithm.

Definition 5.1 (Equidistribution mod m)**.** An integer sequence x_n is said to be equidistributed mod m if and only if

$$\lim_{N\to\infty} \frac{|\{x_i \equiv j \mod m \mid i = 1, ..., N\}|}{N} = \frac{1}{m},$$

for $j = 0, ..., m-1$.

Definition 5.2 (Equidistribution in $\mathbb{Z}_p$)**.** An integer sequence x_n is said to be equidistributed in $\mathbb{Z}_p$ if it is equidistributed mod p^k for every positive integer k.

Here is a proposition as noted by Niven (See Theorem 5.1 of [3]).

Proposition 5.3. Let x_n an integer sequence. If x_n is equidistributed mod p^k then x_n is equidistributed mod p^r for $r = 1, 2, ..., k-1$. Moreover, if x_n is equidistributed mod m then x_n is equidistributed modulo every divisor of m.

Niven actually gives an additional statement in his theorem - If k is not a divisor of m then there exists a sequence that is uniformly distributed mod m but not mod k. He gives an explicit construction of such a sequence (see [3]).

Example 5.4 (Fibonacci Sequence)**.** Let's look at the equidistribution of the Fibonacci sequence $F_0 = 1, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Then

$$F_n = \{1, 1, 2, 3, 5, 8, 13, 21, 34, ...\}.$$

Considering this mod 2 we get

$$F_n \mod 2 = \{1, 1, 0, 1, 1, 0, 1, 1, 0, ...\},$$

since odd + odd is even and even + odd is odd. Clearly, F_n is not equidistributed mod 2, and hence not equidistributed mod 2^k for any k by Proposition 5.3.

Let $D(a, 1/p^k) = a + p^k\mathbb{Z}_p$. Definition 5.2 is equivalent to the following criteria for equidistribution in $\mathbb{Z}_p$

Proposition 5.5. An interger sequence $\{x_n\}$ is equidistributed in $\mathbb{Z}_p$ if and only if for every a in $\mathbb{Z}_p$ and every $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty} \left| \frac{\left|D(a, 1/p^k) \cap \{x_1, ..., x_N\}\right|}{N} - \frac{1}{p^k} \right| = 0.$$

This motivates a definition of discrepancy as follows

Definition 5.6. The discrepancy of a finite integer sequence $\{x_1, x_2, ..., x_N\}$ in $\mathbb{Z}_p$ is

$$D_N = \sup_{a \in \mathbb{Z}_p,\, k \in \mathbb{N}} \left| \frac{\left| D(a, 1/p^k) \cap \{x_1, ..., x_N\} \right|}{N} - \frac{1}{p^k} \right|.$$

Some elementary arguments show that

$$\frac{1}{N} \le D_N \le 1.$$

5.2. **List of Questions and Conjectures.**

(1) **Conjecture:** Show that the discrepancy of the Fibonacci sequence in $\mathbb{Z}_5$ is $O(1/N)$, which is the best possible.

(2) Derive a general lower bound estimate for the discrepancy D_N of a sequence x_n in $\mathbb{Z}_p$.

(3) Hellekalek in [1, 2], studied equidistribution modulo 1 using p-adic arithmetic. In particular, in [1] he derives a general discrepancy bound for sequences modulo 1 using the character group of $\mathbb{Z}_p$. Moreover, it is shown in [2] that an integer sequence $f(n)$ in $\mathbb{Z}_p$ is equidistributed if and only if the sequence $x_{f(n)}$ is equidistributed modulo 1, where x_n is the van der Corput sequence and $x_{f(n)}$ is the subsequence indexed by $f(n)$. Therefore, it would be interesting to study how discrepancy bounds for integer sequences in $\mathbb{Z}_p$ would be related to corresponding subsequences of van der Corput sequences modulo 1.

(4) (O'Dorney) Let $\{a_n\}$ be a sequence of integers satisfying a linear recurrence with constant coefficients, and let p be a prime. Assume that things are generic enough (e.g. the sequence is not periodic, and the constant term of the recurrence is a p-adic unit). Can one show that $\{a_n\}$ equidistributes with respect to a certain measure that is locally constant almost everywhere? Can one also show that the discrepancy is best possible, namely $O(1/N)$?

References

[1] P. Hellekalek, *A general discrepancy estimate based on p-adic arithmetics*, Acta Arith., 139 (2009), pp. 117–129.

[2] P. Hellekalek and H. Niederreiter, *Constructions of uniformly distributed sequences using the* **b**-*adic method*, Unif. Distrib. Theory, 6 (2011), pp. 185–200.

[3] I. Niven, *Uniform distribution of sequences of integers*, Trans. Amer. Math. Soc., 98 (1961), pp. 52–61.

6. Preperiodic points of morphisms over finitely generated fields (Tucker)

Posed by: Thomas J. Tucker, University of Rochester

Question 6.1. *Let X be a quasi-projective variety and let $f : X \longrightarrow X$ be a finite morphism and let K be a finitely generated field over which f is defined. Let Prep(f) be the preperiodic points of f. How generally can we say that Prep$(f) \cap X(K)$ is finite?*

Here are two possible formulations. In each case, for $m > 0$ and $n \geq 0$, we let $X_{m,n}$ be the set of $x \in X$ such that $f^{m+n}(x) = f^n(x)$.

(1) Suppose that $X_{m,n}$ is finite for any m, n. Is $\mathrm{Prep}(f) \cap X(K)$ is necessarily finite?

(2) More generally, let $\mathrm{Prep}^*(f)$ be the set of preperiodic points that do not lie in a positive dimensional component of $X_{m,n}$ for any m, n. Is $\mathrm{Prep}^*(f)$ necessarily finite?

7. Elliptical vs polygonal billiards (Zannier)

Posed by: Umberto Zannier, Scuola Normale Superiore, Pisa

The following is true on an elliptical billiard: "For each fixed angle $\alpha \in (0, \pi/2)$ and a given point P on the boundary, there are only finitely many pairs of periodic billiard trajectories through P forming an angle α at P."

Question 7.1. *For which* ***polygonal*** *billiards is the above statement true? For instance it may be shown that it is not generally true on parallelogram-billiards (though it is true for certain ones of them).*